MATH 1700: SECTION 10.1: RADIAN MEASURE

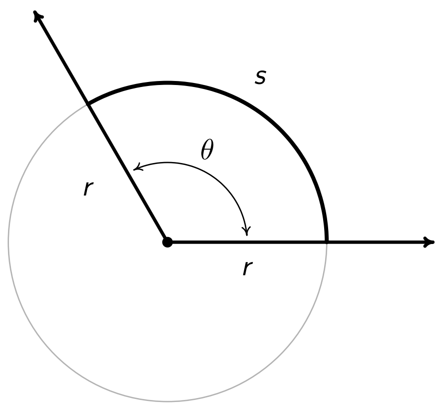
DEFINITION OF RADIAN MEASURE:

RECALL: The real number π is defined to be the ratio of a circle's circumference to its diameter. In symbols, given a circle of circumference C and diameter d,

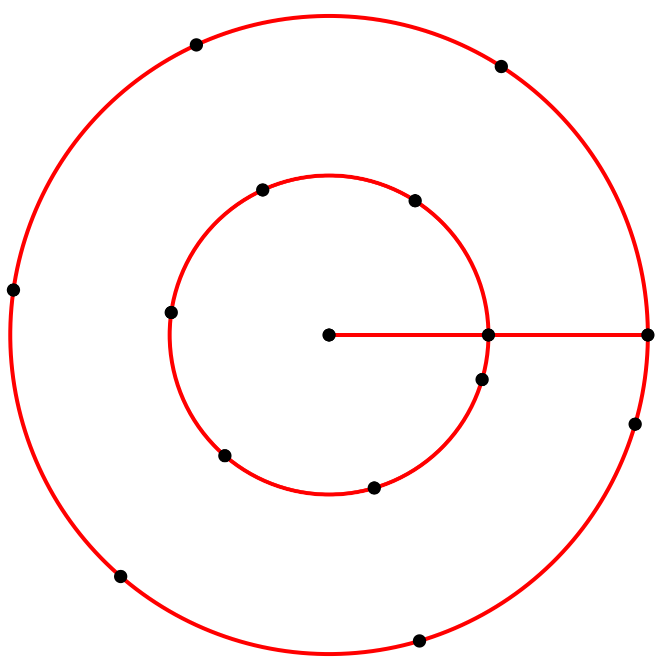
$$\pi = \frac{C}{d} \quad \text{or} \quad C = \pi\, d$$

Since the diameter of a circle is twice the radius, we get $C = 2\pi\, r$.

Suppose we take a *portion* of the circle as depicted below, and we compare some arc measuring s units in length to the radius. Let θ be the **central angle** subtended by this arc. The ratio $\frac{s}{r}$ is the **radian measure** of θ.



Measuring an angle in 'radians' gives the number of 'radius lengths' the angle sweeps out along the circumference:

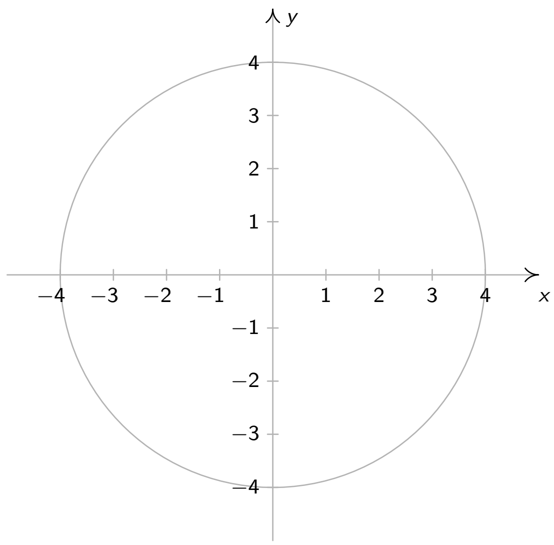


Since one revolution sweeps out the circumference $2\pi r$, one revolution has radian measure $\frac{2\pi r}{r} = 2\pi$. Hence, half of a revolution has radian measure $\frac{1}{2}(2\pi) = \pi$, a quarter revolution has radian measure $\frac{1}{4}(2\pi) = \frac{\pi}{2}$, and so forth. Note that, by definition, the radian measure of an angle is a length divided by another length so that these measurements are actually dimensionless and are considered 'pure' numbers. For this reason, we do not use any symbols to denote radian measure, but we use the word 'radians' to denote these dimensionless units as needed. For instance, we say one revolution measures '2π radians,' half of a revolution measures 'π radians,' and so forth.

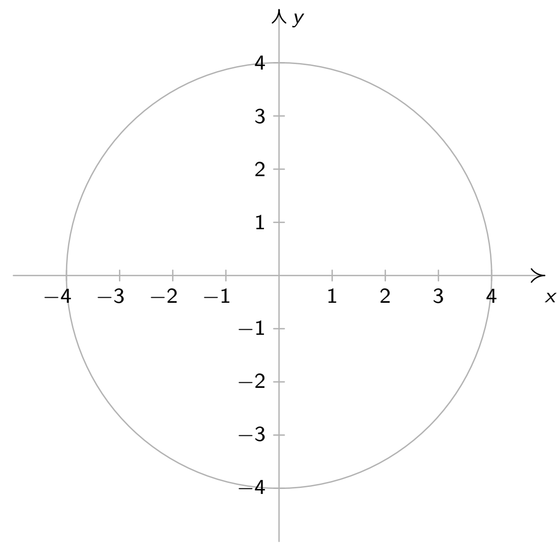
As with degree measure, the distinction between the angle itself and its measure is often blurred in practice, so when we write '$\theta = \frac{\pi}{2}$', we mean θ is an angle which measures $\frac{\pi}{2}$ radians. We extend radian measure to oriented angles, just as we did with degrees beforehand, so that a positive measure indicates counter-clockwise rotation and a negative measure indicates clockwise rotation. Much like before, two positive angles α and β are supplementary if $\alpha + \beta = \pi$ and complementary if $\alpha + \beta = \frac{\pi}{2}$. Finally, we leave it to the reader to show that when using radian measure, two angles α and β are coterminal if and only if $\beta = \alpha + 2\pi k$ for some integer k.

EXAMPLE 1: Graph each of the angles below in standard position and classify them according to where their terminal side lies. Find three coterminal angles, at least one of which is positive and one of which is negative.

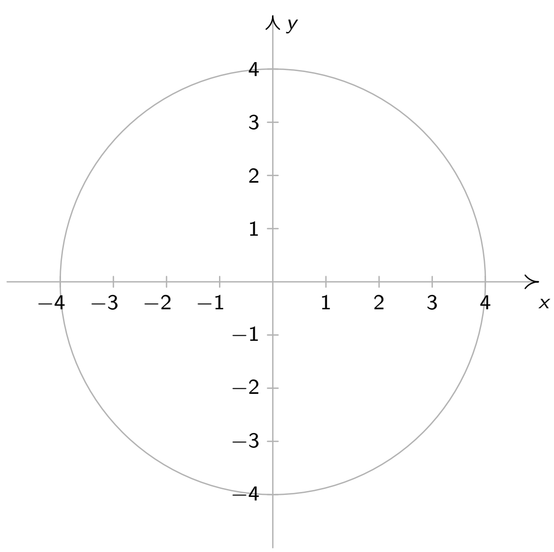
- $\alpha = \dfrac{\pi}{6}$



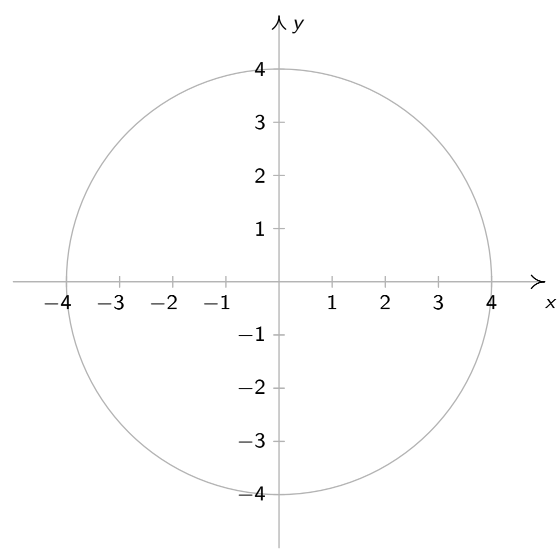
- $\beta = -\dfrac{4\pi}{3}$



- $\gamma = \dfrac{9\pi}{4}$



- $\phi = -\dfrac{5\pi}{2}$



CONVERSION BETWEEN DEGREES AND RADIANS:

- To convert degree measure to radian measure, multiply by $\dfrac{\pi \text{ radians}}{180^\circ}$
- To convert radian measure to degree measure, multiply by $\dfrac{180^\circ}{\pi \text{ radians}}$

EXAMPLE 2:

1. Convert the following angles from degree measure to radian measure:

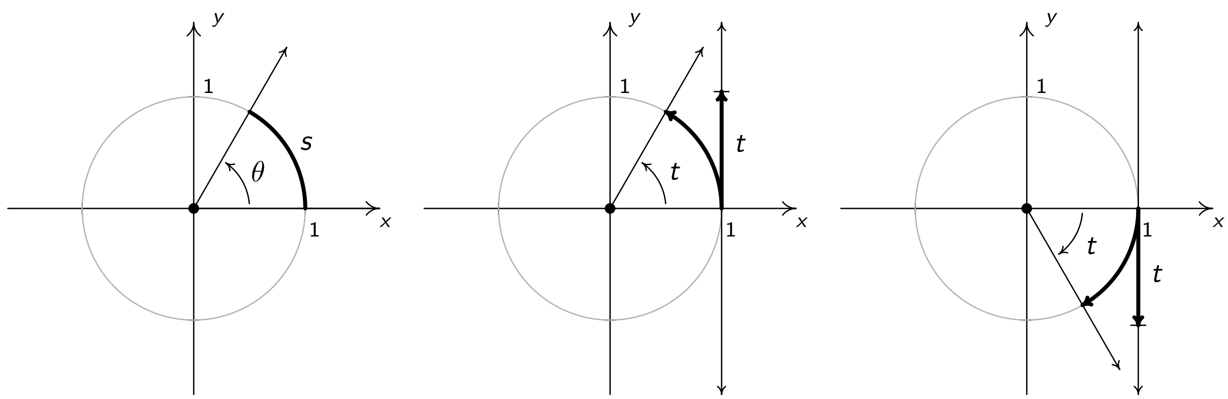
   (a) $\theta = 45^\circ$ (b) $\theta = 210^\circ$ (c) $\theta = -90^\circ$ (d) $\theta = 540^\circ$

2. Convert the following angles from radian measure to degree measure:

   (a) $\theta = \dfrac{\pi}{3}$ (b) $\theta = \dfrac{7\pi}{4}$ (c) $\theta = -\dfrac{5\pi}{6}$ (d) $\theta = \dfrac{7\pi}{2}$

THE 'WRAPPING' FUNCTION: MATCHING ANGLES WITH REAL NUMBERS

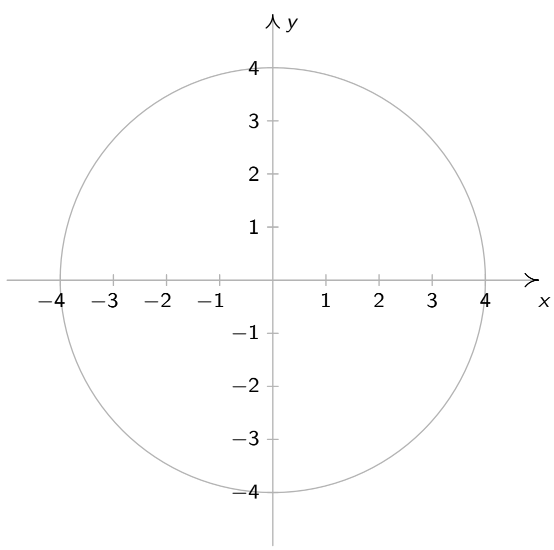
Viewing the vertical line $x = 1$ as another real number line demarcated like the y-axis, given a real number $t > 0$, we 'wrap' the (vertical) interval $[0, t]$ around the Unit Circle in a counter-clockwise fashion. The resulting arc has a length of t units and therefore the corresponding angle has radian measure equal to t. If $t < 0$, we wrap the interval $[t, 0]$ *clockwise* around the Unit Circle. Since we have defined clockwise rotation as having negative radian measure, the angle determined by this arc has radian measure equal to t. If $t = 0$, we are at the point $(1, 0)$ on the x-axis which corresponds to an angle with radian measure 0. In this way, we identify each real number t with the corresponding angle with radian measure t.



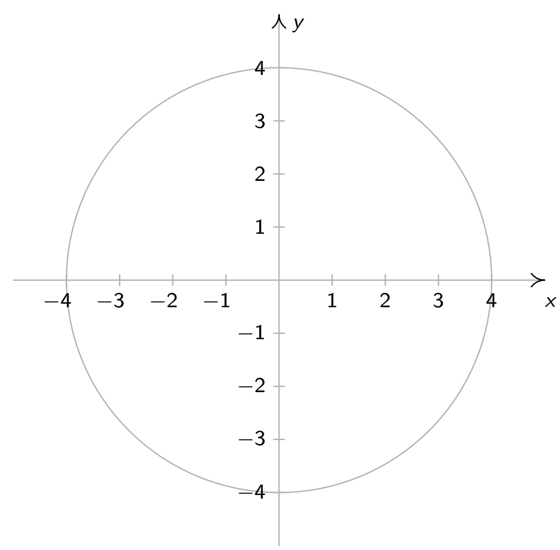
On the Unit Circle, $\theta = s$. Identifying $t > 0$ with an angle. Identifying $t < 0$ with an angle.

EXAMPLE 3: Sketch the oriented arc on the Unit Circle corresponding to each of the following real numbers.

- $t = \dfrac{3\pi}{4}$

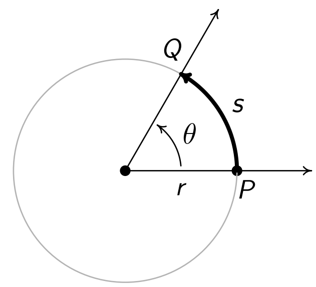


- $t = -2$



APPLICATIONS OF RADIAN MEASURE: CIRCULAR MOTION

Suppose an object is moving as pictured below along a circular path of radius r from the point P to the point Q in an amount of time t.



Here s represents a *displacement* so that $s > 0$ means the object is traveling in a counter-clockwise direction and $s < 0$ indicates movement in a clockwise direction. Note that with this convention the formula we used to define radian measure, namely $\theta = \frac{s}{r}$, still holds since a negative value of s incurred from a clockwise displacement matches the negative we assign to θ for a clockwise rotation. In Physics, the **average velocity** of the object, denoted $\overline{v}$ and read as 'v-bar', is defined as the average rate of change of the position of the object with respect to time.As a result, we have $\overline{v} = \frac{\text{displacement}}{\text{time}} = \frac{s}{t}$. The quantity $\overline{v}$ has units of $\frac{\text{length}}{\text{time}}$ and conveys two ideas: the direction in which the object is moving and how fast the position of the object is changing. The contribution of direction in the quantity $\overline{v}$ is either to make it positive (in the case of counter-clockwise motion) or negative (in the case of clockwise motion), so that the quantity $|\overline{v}|$ quantifies how fast the object is moving - it is the **speed** of the object. Measuring θ in radians we have $\theta = \frac{s}{r}$ thus $s = r\theta$ and

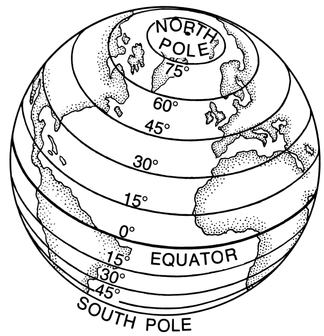
$$\overline{v} = \frac{s}{t} = \frac{r\theta}{t} = r \cdot \frac{\theta}{t}$$

The quantity $\frac{\theta}{t}$ is called the **average angular velocity** of the object. It is denoted by $\overline{\omega}$ and is read 'omega-bar'. The quantity $\overline{\omega}$ is the average rate of change of the angle θ with respect to time and thus has units $\frac{\text{radians}}{\text{time}}$. If $\overline{\omega}$ is constant throughout the duration of the motion, then it can be shown that the average velocities involved, namely $\overline{v}$ and $\overline{\omega}$, are the same as their instantaneous counterparts, v and ω, respectively. In this case, v is simply called the 'velocity' of the object and ω is called the 'angular velocity.'

If the path of the object were 'uncurled' from a circle to form a line segment, then the velocity of the object on that line segment would be the same as the velocity on the circle. For this reason, the quantity v is often called the *linear* velocity of the object in order to distinguish it from the *angular* velocity, ω. Putting together the ideas of the previous paragraph, we get the following.

VELOCITY FOR CIRCULAR MOTION: For an object moving on a circular path of radius r with constant angular velocity ω, the (linear) velocity of the object is given by $v = r\omega$.

EXAMPLE 4: Assuming that the surface of the Earth is a sphere, any point on the Earth can be thought of as an object traveling on a circle (this is the **parallel of latitude** of the point) as seen in the figure below.[1] Since it takes the Earth (approximately) 24 hours to rotate, the object takes 24 hours to complete one revolution along this circle. Lakeland Community College is at 41.628° north latitude, and it can be shown that the radius of the earth at this Latitude is approximately 2960 miles. Find the linear velocity, in miles per hour, of Lakeland Community College as the world turns.



It is worth noting that the quantity $\frac{1 \text{ revolution}}{24 \text{ hours}}$ is called the **ordinary frequency** of the motion and is usually denoted by the variable f. The ordinary frequency is a measure of how often an object makes a complete cycle of the motion. The fact that $\omega = 2\pi f$ suggests that ω is also a frequency. Indeed, it is called the **angular frequency** of the motion. On a related note, the quantity $T = \frac{1}{f}$ is called the **period** of the motion and is the amount of time it takes for the object to complete one cycle of the motion. In our scenario above, the period of the motion is 24 hours, or one day.

The concepts of frequency and period help frame the equation $v = r\omega$ in a new light. That is, if ω is fixed, points which are farther from the center of rotation need to travel faster to maintain the same angular frequency since they have farther to travel to make one revolution in one period's time. The distance of the object to the center of rotation is the radius of the circle, r, and is the 'magnification factor' which relates ω and v.

HOMEWORK: Section 10.1: Page 816: 1 - 37 odd, 38, 39, 45, 49, 51*

[1]Diagram credit: Pearson Scott Foresman [Public domain], via Wikimedia Commons.